Int. J. Solids Structures Vol. 29, No. 2, pp. 249–270, 1992
Printed in Great Britain.

0020-7683/92 $3.00+ .00

EXTREMUM, CONVERGENCE AND STABILITY PROPERTIES OF THE FINITE-INCREMENT PROBLEM IN ELASTIC–PLASTIC BOUNDARY ELEMENT ANALYSIS

CLAUDIA COMI and GIULIO MAIER
Department of Structural Engineering, Politecnico of Milan, 20133 Milan, Italy

(Received 24 January 1991)

Abstract—The boundary element (BE) analysis is formulated by a symmetric (Galerkin weighted-residual, double-integration) approach, rather than by a traditional collocation or by a non-symmetric-Galerkin approach. The internal variable associative elastoplastic material model is discretized in time by a stepwise-holonomic, backward-difference integration scheme; it is then enforced in a weighted-average sense over cells and reformulated in terms of cell generalized variables.

In the above context the following results are established under suitable constitutive hypotheses: (a) a minimum characterization of the solution to the discretized step-problem in finite increments; (b) a convergence theorem concerning a conventional iterative algorithm for solving this problem; (c) a proof of the stability of the marching solution method, in the sense of non-amplification of errors along a finite step sequence. An illustrative example corroborates the theoretical results.

NOTATION

Bold face symbols denote matrices and column vectors. **0** is a vector whose entries are all zero. Inequalities apply componentwise. Superscript t means transpose, a dot denotes time derivative. In order to remove possible ambiguity between argument of a function and multiplication, in the latter case a dot will precede the parenthesis. Other symbols are defined where they are used for the first time.

INTRODUCTION

The traditional boundary element (BE) methods [see e.g. Banerjee and Butterfield (1981); Mukherjee (1982); Brebbia *et al.* (1984)] are centered on integral operators and (after discretization) matrix operators, which do not exhibit properties (such as self-adjointness or symmetry and sign-definiteness) leading to meaningful and useful theoretical conclusions available in other formulations and solution methods. A kind of widespread dissatisfaction due to this fact led to various "symmetrization" procedures. Confining ourselves to the quasi-static elastoplasticity of concern in this paper, we mention below earlier proposals and developments which appear to be somehow related to the present results.

Algebraic "forced" symmetrization of matrix operators (specifically stiffness matrices) arising from conventional collocation approaches was put forward and advocated by some authors in elasticity [e.g. Zienkiewicz *et al.* (1977)]. It was extended to plasticity by Maier (1983) and Maier and Nappi (1984), who showed how "symmetrized" BE formulations preserve the validity of some meaningful aspects of plasticity theory (extremum characterizations of incremental solutions, shakedown and bounding theorems).

The Galerkin, double-integration approach leading to symmetric BE formulations, proposed first for linear-elastic problems by Sirtori (1979), has been developed in elastoplasticity by Maier and Polizzotto (1987), Polizzotto (1988) and Maier *et al.* (1989, 1990) and provides the basis for the present contributions. Recent alternative symmetrizations pointed out by Teixeira de Freitas (1990) and Bui (1990) exhibit interesting features, but their computational implications and extensions to plasticity are still to be investigated.

In computational plasticity centered on finite element (FE) discretizations in space, much attention has been paid in recent years to step-by-step marching solution methods. This research topic has been vigorously tackled particularly by Martin and his co-workers [see e.g. Martin *et al.* (1987); Caddemi and Martin (1991); Ortiz and Martin (1989)], with reference to internal variable descriptions of the plastic material behaviour. Among other important contributions are those by Krieg and Krieg (1977), Ortiz and Popov (1985), Simo *et al.* (1988) and Perego (1988). Earlier work in this area of stepwise-holonomic elastoplastic analysis was based on the piecewise linearization of the yield surface, and centered on extremum properties of step solutions and quadratic programming concepts and algorithms [see representative paper by De Donato and Maier (1972) and surveys by Maier and Munro (1982)]. It appears that limited interaction has occurred so far between the above mentioned developments in computational plasticity by FE and those in the traditional BE context [see e.g. Brebbia *et al.* (1984); Cruse and Polch (1986)].

In this paper a contribution is made to BE elastic–plastic analysis in directions, in a sense, parallel to those in which progress was recently achieved in FE inelastic analysis as mentioned above.

The material behaviour is described herein by a fairly general, internal variable elastic–plastic constitutive law, which is integrated in time by a backward-difference scheme. Discretized boundary integral equations, such that symmetry of their coefficient matrices may be guaranteed, are generated by the following provisions as proposed by Maier and Polizzotto (1987): use of static and kinematic sources on the boundary; space discretization by a Galerkin weighted-residual approach; "consistent" modelling of domain unknown fields by suitable "generalized variables" for cells; weighted average (instead of pointwise) enforcement of the material model. On this basis, the findings presented herein concerning the discretized step-problem in finite increments are as follows: (a) Sufficient and necessary conditions, in terms of optimization problems, for the solution to the finite step elastic–plastic problem; these results are related to various extremum characterizations of solutions to finite-step boundary value problems recently established by Comi *et al.* (1991a). (b) A proof of convergence of an iterative, predictor–corrector, successive substitution algorithm for solving this problem; a basically similar path of reasoning led Comi and Maier (1990) to an analogous result in the FE context. (c) A proof of the stability of the proposed marching, step-by-step solution procedure; here stability means the lack of error amplification along a sequence of loading steps, in the sense of the non-linear "B-stability" of Simo (1991) and Simo and Govindjee (1991).

The theoretical results expounded are corroborated by numerical tests carried out using a two-dimensional implementation of the symmetric BE method presented in detail in Maier *et al.* (1991).

GOVERNING RELATIONS

We refer to a homogeneous elastic–plastic solid or structure which occupies the volume Ω (conceived as an open domain) with the boundary Γ. Under the hypothesis of small deformations ("geometric" linearity) the response of this solid is sought to a given history of the following actions: displacements $\bar{u}_i(t)$ on the constrained portion Γ_u or Γ; tractions $\bar{p}_i(t)$ on the complementary portion Γ_p; body forces $\bar{b}_i(t)$ and imposed "initial" (such as thermal) strains $\bar{\theta}_{ij}(t)$ in Ω. The external forces are assumed to be conservative. A Cartesian reference and the index summation convention are adopted. Commas will denote space derivatives; dots time derivatives. "Time" t represents any variable which monotonically increases in the physical time and merely orders events; this is equivalent to stating that the mechanical phenomena to study are time-independent or "inviscid".

The quasi-static evolution of the considered solid from the original state (say the unstressed state at $t = 0$) is governed by the set of relations:

$$\sigma_{ij,j} + \bar{b}_i = 0 \quad \text{in } \Omega, \quad \sigma_{ij} n_j = \bar{p}_i \quad \text{on } \Gamma_p \tag{1}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \Omega, \quad u_i = \bar{u}_i \quad \text{on } \Gamma_u \tag{2}$$

$$\varepsilon_{ij} = e_{ij} + \bar{\theta}_{ij} + \varepsilon^p_{ij}, \quad e_{ij} = C_{ijhk}\sigma_{hk} \tag{3}$$

$$\phi(\sigma_{ij}, q_h) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \phi\dot{\lambda} = 0 \tag{4}$$

$$\dot{\varepsilon}^p_{ij} = \frac{\partial \psi}{\partial \sigma_{ij}}(\sigma_{rs}, q_h)\dot{\lambda}, \quad \dot{\eta}_h = -\frac{\partial \psi}{\partial q_h}(\sigma_{ij}, q_k)\dot{\lambda} \tag{5}$$

$$q_h = \frac{\partial w}{\partial \eta_h}(\eta_k) \tag{6}$$

$$\dot{D} = \sigma_{ij}\dot{\varepsilon}^p_{ij} - q_h\dot{\eta}_h \geqslant 0. \tag{7}$$

Here, by a customary notation: eqns (1) express equilibrium (n_i being the unit outward normal to Γ, which is assumed to be "smooth" for simplicity); eqns (2) enforce geometric compatibility; eqns (3a, b) reflect the strain additivity and Hooke's law (C_{ijhk} denoting the positive-definite elastic compliance tensor with the usual symmetries and ε^p_{ij} plastic strains). Equations (4) formulate the yield criterion by means of a single differentiable yield function ϕ (no "corners" for the sake of simplicity in further developments); eqns (5) express the generalized flow rule of the plastic material model (associative if $\psi = \phi$). We denote by q_h ($h = 1, \ldots, n_y$) the static internal variables and by η_h the conjugate kinematic internal variables [see e.g. Halphen and Nguyen (1975)]. If $n_y = 0$ the constitutive law specializes to ideal plasticity. The vector notation adopted here for internal variables does not mean that the possibility of their tensorial nature (say q_{ij}) is ruled out. In eqn (6) which relates static to kinematic internal variables, w can be interpreted as the "stored" free energy due to structural rearrangements at the microscale. Inequality (7) expresses the thermodynamic requirement at the rate of dissipation $\dot{D}$.

CONSTITUTIVE RESTRICTIONS

We list below the further assumptions which are adopted for the class of constitutive models described by eqns (4)–(7) and which will be used later in the paper in order to establish extremum, convergence and stability properties.

(a) The plastic strain and kinematic internal variable rates ($\dot{\varepsilon}^p_{ij}$ and $\dot{\eta}_h$) are "associated" to the current yield locus defined by eqn (4a); in other words $\psi = \phi$ in eqns (5).

(b) The yield function ϕ is convex in both stresses σ_{ij} and static internal variables q_h.

(c) The potential w of the internal variables (i.e. the "stored" energy density) is a convex function of η_h.

(d) The yield function ϕ is expressed as the difference of two terms: an effective stress f and a constant yield limit y; the former addend is a positively homogeneous function of order one of the stresses and the static internal variables.

It is worth noting that assumptions (a)–(c) imply the validity of Drucker's stability postulate, Drucker (1951). In fact, normality of plastic strain rates and convexity of yield surfaces are consequences of (a) and (b), which exclude frictional materials with non-associative flow rules; hypothesis (c) could easily be seen to imply that the second order plastic work cannot be negative ($\frac{1}{2}\delta\sigma_{ij}\delta\varepsilon^p_{ij} \geqslant 0$), thus ruling out softening behaviour. It might also be shown that a multiplicity of yield modes implying "corners" of yield surface would preserve the above essential features of material stable behaviour and that, in this case, the existence of the potential $w(\eta_h)$ implies "reciprocal hardening" between yielding modes, Comi *et al.* (1991a).

Hypotheses (a)–(c) represent physically meaningful restrictions on the coverage of eqns (4)–(7), which thus become a description of the still broad category of so-called "generalized standard elastoplastic materials" (Halphen and Nguyen, 1975) with a single yield mode. On the contrary, hypothesis (d) entails no loss of generality, since it might easily be shown to reflect an always possible equivalent way of representing yield criteria. As a consequence of (d), by Euler's theorem,

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial q_h} \dot{q}_h;$$

this expression will be used below in eqn (8), noting that the derivatives of f and ϕ coincide, since f and ϕ differ by the constant y.

DISCRETIZATION IN TIME

For the approximate time integration of the differential relation set (1)–(6), we will adopt the following traditional strategy. Consider a monotonic sequence of instants $t_0 = 0$, $t_1, \ldots, t_n, t_{n+1} = t_n + \Delta t$. Let all variables be known at t_n and mark them by barred symbols. The increments (marked by Δ) of the unknowns are sought for the given increments over Δt of the external actions, namely for given $\Delta \bar{b}_i$, $\Delta \bar{\theta}_{ij}$, $\Delta \bar{p}_i$, $\Delta \bar{u}_i$.

This problem is referred to henceforth as "finite-step" elastoplastic boundary value problem. It concerns finite increments and it is path-independent (or holonomic) over the relevant time interval Δt. In other terms, the exact time integration of the non-linear differential governing relation set (1)–(6) will be approximated by a "stepwise-holonomic" analysis.

While the linear eqns (1)–(3) can be directly rewritten in terms of increments, the non-linear constitutive relations (4)–(6) must be algebrized according to some approximation scheme. Various schemes have been proposed and investigated in the literature, based on diverse hypotheses of the yielding process over Δt: forward Euler scheme; generalized trapezoidal rule; generalized mid-point rule; backward-difference method [see e.g. Ortiz and Popov (1985); Simo *et al.* (1988)]. We choose here the backward-difference method which was proposed and applied in earlier works to elastoplastic analysis by quadratic programming (De Donato and Maier, 1972) and extensively studied recently in more general contexts by Martin and his co-workers (Martin *et al.*, 1987). For the present constitutive models the backward-difference concept materializes in the following approximate algebraic version of eqns (4)–(6):

$$\phi = \frac{\partial \phi}{\partial \sigma_{ij}}(\bar{\sigma}_{rs} + \Delta\sigma_{rs}, \bar{q}_k + \Delta q_k) \cdot (\bar{\sigma}_{ij} + \Delta\sigma_{ij}) + \frac{\partial \phi}{\partial q_h}(\bar{\sigma}_{rs} + \Delta\sigma_{rs}, \bar{q}_k + \Delta q_k) \cdot (\bar{q}_h + \Delta q_h) - y \leqslant 0 \quad (8)$$

$$\Delta\varepsilon^p_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}}(\bar{\sigma}_{rs} + \Delta\sigma_{rs}, \bar{q}_k + \Delta q_k)\Delta\lambda, \quad \Delta\eta_h = -\frac{\partial \phi}{\partial q_h}(\bar{\sigma}_{rs} + \Delta\sigma_{rs}, \bar{q}_k + \Delta q_k)\Delta\lambda \quad (9)$$

$$\bar{q}_h + \Delta q_h = \frac{\partial w}{\partial \eta_h}(\eta_k); \quad \Delta\lambda \geqslant 0, \quad \phi\Delta\lambda = 0. \quad (10)$$

Equations (8)–(10), together with eqns (1)–(3) rephrased in increments, form a relation set which governs the finite-step b.v. problem to be discretized in space below and discussed in the sequel.

INTEGRAL EQUATIONS FOR ELASTIC PLASTIC ANALYSIS

For the sake of brevity, henceforth we will adopt matrix notation and assume zero body forces and, later, zero initial strains (body forces $\mathbf{b}$ would merely imply self-evident extra addends in eqn (11) and in its consequences). Thus $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\theta}$ will denote vectors of the independent components of stress, total strain and inelastic or imposed strain tensors, respectively (with the "engineering definition" of shear strains). The index sum convention no longer holds.

Consider the linear elastic b.v. problem governed by eqns (1)–(3) with $\varepsilon^p_{ij} = 0$. Unstarred symbols will denote the solution to it for the given external actions; starred symbols mark a "fictitious" auxiliary elastic state, i.e. a solution to the problem for suitably chosen fictitious external actions. Then Betti's reciprocity theorem of linear elasticity can be expressed by the equation:

$$\int_\Gamma (\mathbf{p}^t\mathbf{u}^* - \mathbf{u}^t\mathbf{p}^*)\,\mathrm{d}\Gamma + \int_\Omega \boldsymbol{\theta}^t\boldsymbol{\sigma}^*\,\mathrm{d}\Omega = \int_\Omega \boldsymbol{\sigma}^t\boldsymbol{\theta}^*\,\mathrm{d}\Omega + \int_\Omega \mathbf{b}^{*t}\mathbf{u}\,\mathrm{d}\Omega. \quad (11)$$

The traditional choice is to identify the starred elastic state with Kelvin's fundamental solution (or with a half-space solution: Mindlin's in three-, Melan's in two-dimensional cases). This leads to the customary integral representation of displacements (Somigliana identity) and to the consequent boundary integral equation with a non-symmetric (non self-adjoint) integral operator [see e.g. Banerjee and Butterfield (1981) and Brebbia *et al.* (1984)].

Instead we will follow the approach proposed by Sirtori (1979) in elasticity and by Maier and Polizzotto (1987) in elastoplasticity, in order to generate symmetric operators. To this purpose, let us choose as fictitious elastic state in eqn (11), the response of the elastic space Ω_∞ (embedding Ω) to distributions of surface forces $\mathbf{F}^*$ and displacement discontinuities $\mathbf{D}^*$ on the boundary Γ and of imposed strains $\boldsymbol{\theta}^*$ on the volume Ω. Such a linear elastic response by Ω_∞ in terms of displacements, tractions and stresses can be expressed by superposition of effects using the matrices of Green's functions for Ω_∞ denoted by $\mathbf{G}_{hk}$ ($h, k = u, p, \sigma$):

$$\underset{\mathbf{x}\in\Gamma}{\mathbf{u}^*(\mathbf{x})} = \int_\Gamma \mathbf{G}_{uu}(\mathbf{x},\boldsymbol{\xi})\mathbf{F}^*(\boldsymbol{\xi})\,\mathrm{d}\Gamma + \int_\Gamma \mathbf{G}_{up}(\mathbf{x},\boldsymbol{\xi})\mathbf{D}^*(\boldsymbol{\xi})\,\mathrm{d}\Gamma + \int_\Omega \mathbf{G}_{u\sigma}(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\theta}^*(\boldsymbol{\xi})\,\mathrm{d}\Omega \quad (12a)$$

$$\underset{\mathbf{x}\in\Gamma}{\mathbf{p}^*(\mathbf{x})} = \int_\Gamma \mathbf{G}_{pu}(\mathbf{x},\boldsymbol{\xi})\mathbf{F}^*(\boldsymbol{\xi})\,\mathrm{d}\Gamma + \int_\Gamma \mathbf{G}_{pp}(\mathbf{x},\boldsymbol{\xi})\mathbf{D}^*(\boldsymbol{\xi})\,\mathrm{d}\Gamma + \int_\Omega \mathbf{G}_{p\sigma}(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\theta}^*(\boldsymbol{\xi})\,\mathrm{d}\Omega \quad (12b)$$

$$\underset{\mathbf{x}\in\Omega}{\boldsymbol{\sigma}^*(\mathbf{x})} = \int_\Gamma \mathbf{G}_{\sigma u}(\mathbf{x},\boldsymbol{\xi})\mathbf{F}^*(\boldsymbol{\xi})\,\mathrm{d}\Gamma + \int_\Gamma \mathbf{G}_{\sigma p}(\mathbf{x},\boldsymbol{\xi})\mathbf{D}^*(\boldsymbol{\xi})\,\mathrm{d}\Gamma + \int_\Omega \mathbf{G}_{\sigma\sigma}(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\theta}^*(\boldsymbol{\xi})\,\mathrm{d}\Omega. \quad (12c)$$

In the kernels of eqns (12), $\mathbf{x} \equiv \{x_i\}$ represents the point where the effect is evaluated (field point), $\boldsymbol{\xi} \equiv \{\xi_\alpha\}$ the point where the unit concentrated source is applied (load or source point). The symbols $\mathbf{x}^+$ or $\boldsymbol{\xi}^+$ and $\mathbf{x}^-$ or $\boldsymbol{\xi}^-$ denote points which are exterior and interior to Ω, respectively, and are infinitely close to points $\mathbf{x}$ or $\boldsymbol{\xi}$ on Γ. By Γ^+ and Γ^- we will denote the surfaces formed by the sets of such points. Thus the static and kinematic discontinuities which generate the auxiliary elastic state (12) can be expressed as:

$$\mathbf{F}^*(\boldsymbol{\xi}) = \mathbf{p}^*(\boldsymbol{\xi}^+) - \mathbf{p}^*(\boldsymbol{\xi}^-), \quad \mathbf{D}^*(\boldsymbol{\xi}) = \mathbf{u}^*(\boldsymbol{\xi}^+) - \mathbf{u}^*(\boldsymbol{\xi}^-). \quad (13)$$

The above Green function matrices $\mathbf{G}_{hk}$ [using a suitable notation proposed by Polizzotto (1988)], have the following meanings.

(a) For $k = u$: displacements ($h = u$), tractions ($h = p$) and stresses ($h = \sigma$) in $\mathbf{x}$ due to a unit concentrated force acting in $\boldsymbol{\xi}$ and directed according to the reference axis $\alpha = 1$, 2, 3 in turn, for the three columns of $\mathbf{G}_{hu}$.

(b) For $k = p$: quantities as above, but due to a concentrated displacement discontinuity across Γ with unit integral over Γ, acting in $\boldsymbol{\xi}$ and directed according to the reference axis $\alpha = 1, 2, 3$ in turn.

(c) For $k = \sigma$: similar quantities as above, now due to a concentrated imposed strain with unit integral over Ω, acting on point $\boldsymbol{\xi}$ and with only one non-zero component $\alpha\beta$, in turn, for the six columns of $\mathbf{G}_{h\sigma}$ in three-dimensional situations (shear strain for $\alpha \neq \beta$, intervenes only once).

Clearly, when the effect is a traction at $\mathbf{x}$ ($h = p$), the outward normal $\mathbf{n}$ to Γ in $\mathbf{x}$ intervenes in the explicit expression of matrix $\mathbf{G}_{pk}$ for $k = u, p, \sigma$.

The above mentioned concentrated unit sources can be described as the following special distributions of traction (static) discontinuities, displacement (kinematic) discontinuities and imposed strains, using the Dirac "function" $\delta(\mathbf{x}-\boldsymbol{\xi})$ and indicating by superscripts the unit component (the others being zero):

$$\mathbf{F}^{\alpha}\delta(\mathbf{x}-\boldsymbol{\xi}),\quad \mathbf{D}^{\alpha}\delta(\mathbf{x}-\boldsymbol{\xi}),\quad \theta^{\alpha\beta}\delta(\mathbf{x}-\boldsymbol{\xi}). \tag{14}$$

Note that the three matrix kernels above referred to in (a) describe the usual Kelvin elastic state and their analytical expressions can be found in any book on BEM. The three kernels (b) should probably be referred to as Gebbia state in view of the extensive but forgotten study conducted by this author about a century ago (Gebbia, 1891). The Green function matrices (c) were given a correct definitive form in Bui (1977) and are used in all recent inelastic analyses by BE.

The three kinds of Green functions are gathered below with specification of their singularity order for $\mathbf{x} = \boldsymbol{\xi}$ in three-dimensional situations:

$$\mathbf{G}_{uu}(r^{-1})\quad \mathbf{G}_{up}(r^{-2})\quad \mathbf{G}_{u\sigma}(r^{-2}) \tag{15a}$$

$$\mathbf{G}_{pu}(r^{-2})\quad \mathbf{G}_{pp}(r^{-3})\quad \mathbf{G}_{p\sigma}(r^{-3}) \tag{15b}$$

$$\mathbf{G}_{\sigma u}(r^{-2})\quad \mathbf{G}_{\sigma p}(r^{-3})\quad \mathbf{G}_{\sigma\sigma}(r^{-3}). \tag{15c}$$

Among the above kernels there are interrelations of two kinds:

(i) those which arise from the very nature of the effect considered in $\mathbf{x}$ and of the source considered in $\boldsymbol{\xi}$; these relationships imply derivatives with respect to $\mathbf{x}$ or $\boldsymbol{\xi}$;

(ii) relationships which are due to Betti's reciprocity theorem could be derived from eqn (11) (Maier and Polizzotto, 1987) and can be expressed, for $\mathbf{x} \neq \boldsymbol{\xi}$, in the following compact form:

$$\mathbf{G}_{hk}(\mathbf{x},\boldsymbol{\xi}) = \mathbf{G}^{\mathrm{t}}_{hk}(\boldsymbol{\xi},\mathbf{x})\quad h,k = u,p,\sigma. \tag{16}$$

A detailed discussion of these relations can be found in Sirtori *et al.* (1991), with emphasis on the special difficulties concerning $\mathbf{G}_{up}$ and the hypersingularities of $\mathbf{G}_{pp}$ for $\mathbf{x} = \boldsymbol{\xi}$.

As usual in BE inelastic analysis, let us subdivide into elements the boundary Γ and into cells the part of domain Ω where yielding is expected.

Using suitable polynomial interpolations contained in shape matrices $\boldsymbol{\Psi}$, we discretize both the actual fields and the source distributions which generate in Ω, the fictitious, auxiliary (starred) fields:

$$\mathbf{p}(\mathbf{x}) = \boldsymbol{\Psi}_p(\mathbf{x})\mathbf{P},\quad \mathbf{u}(\mathbf{x}) = \boldsymbol{\Psi}_u(\mathbf{x})\mathbf{U},\quad \theta(\mathbf{x}) = \boldsymbol{\Psi}_\theta(\mathbf{x})\boldsymbol{\Theta} \tag{17}$$

$$\mathbf{F}^*(\mathbf{x}) = \boldsymbol{\Psi}^*_p(\mathbf{x})\mathbf{F}^{**},\quad \mathbf{D}^*(\mathbf{x}) = \boldsymbol{\Psi}^*_u(\mathbf{x})\mathbf{D}^{**},\quad \theta^*(\mathbf{x}) = \boldsymbol{\Psi}^*_\theta(\mathbf{x})\boldsymbol{\Theta}^{**}. \tag{18}$$

Here vectors $\mathbf{P}$, $\mathbf{U}$, $\boldsymbol{\Theta}$, $\mathbf{F}^{**}$, $\mathbf{D}^{**}$, $\boldsymbol{\Theta}^{**}$ collect the values assumed by the relevant quantities at the nodes. The nodes for field modelling in Ω will be regarded as chosen in its interior, in order to simplify the preliminaries for subsequent developments.

The interpolation matrices $\boldsymbol{\Psi}$ are constructed over each element or cell but are conceived as defined, for each node, over the whole Γ or Ω, as "support functions" of that node, taking into account the possible continuity requirements at interfaces between adjacent boundary elements or cells.

The discretization represented by eqns (17) and (18) shall comply with the following restrictive provisions, whose motivations will become clear later:

(i) The displacements $\mathbf{u}(\mathbf{x})$ and the kinematic discontinuities $\mathbf{D}^*(\mathbf{x})$ are **continuous** across interfaces between adjacent elements on Γ.

(ii) The models for corresponding actual and source fields are equal: $\mathbf{\Psi}_i = \mathbf{\Psi}_i^*$, for $i = u, p, \theta$.

(iii) The shape function pertaining to an **unknown** nodal traction is orthogonal to the shape function relevant to any **unknown** nodal displacement. Clearly, this third requirement is not necessarily satisfied, and becomes meaningful, only for nodes whose "support regions" (where the shape function does not identically vanish) is intersected by the border between the constrained and free portions Γ_u and Γ_p of the boundary.

Let us set:

$$\mathbf{\Sigma} = \int_\Omega \mathbf{\Psi}_\theta^t(\mathbf{x})\sigma(\mathbf{x})\,\mathrm{d}\Omega \tag{19}$$

$$\hat{\mathbf{G}}_{hk} \equiv \iint \mathbf{\Psi}_{h'}^t(\mathbf{x})\mathbf{G}_{hk}(\mathbf{x},\xi)\mathbf{\Psi}_{k'}(\xi)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi;\quad h,k = u,p,\sigma,\quad h',k' = p,u,\theta \tag{20}$$

where h' and k' mark the conjugate of h and k, respectively; $\mathrm{d}\mathbf{x}$ stands for $\mathrm{d}\Gamma(\mathbf{x})$ or $\mathrm{d}\Omega(\mathbf{x})$, $\mathrm{d}\xi$ for $\mathrm{d}\Gamma(\xi)$ or $\mathrm{d}\Omega(\xi)$ and the integration domains are defined by the indices (e.g. if $h = \sigma$, $k = p$, and, hence, $h' = \theta$, $k' = u$: $\mathbf{x}$ runs over Ω and ξ over Γ_p).

As a consequence of the reciprocity eqns (16) and of the discussion of the case $\xi = \mathbf{x}$ developed in Sirtori *et al.* (1991) and not repeated here, one may write (δ_{ij} denoting the Kronecker symbol; no index sum convention):

$$\hat{\mathbf{G}}_{hk} = \hat{\mathbf{G}}_{kh}^t - \int_\Gamma \mathbf{\Psi}_u^t\mathbf{\Psi}_p\,\mathrm{d}\Gamma\,\delta_{hu}\delta_{kp}\quad \text{for } h,k = u,p,\sigma. \tag{21}$$

The above restriction (i) ensures that the double integrations (20) have a meaning and lead to finite integrals even in the presence of the hypersingularities for $\mathbf{x} = \xi$ (Sirtori *et al.*, 1991). In fact, the hypersingular integrals can be interpreted as work associated with a suitably loaded (or "pressurized") crack in Ω_r. Note that all kernels involving subscript σ do not exhibit singularities in integrations (20) since $\mathbf{x} \neq \xi$, except $\mathbf{G}_{\sigma\sigma}$. However, the improper domain integral containing $\mathbf{G}_{\sigma\sigma}$ is a usual ingredient of traditional BE plastic analysis and, therefore, is not discussed here [see Bui (1977)].

We now introduce the discretization (17) and (18) into eqns (11), taking account of the second (ii) of the above modelling restrictions. Substituting thereafter eqns (12) into (11), using the definitions (19) and (20), after trivial manipulations the Betti eqn (11) can be given the form:

$$\begin{Bmatrix} \mathbf{F}^{**} \\ -\mathbf{D}^{**} \\ \mathbf{\Theta}^{**} \end{Bmatrix}^t \left[\begin{bmatrix} \hat{\mathbf{G}}_{uu} & -\hat{\mathbf{G}}_{up} & \hat{\mathbf{G}}_{u\sigma} \\ -\hat{\mathbf{G}}_{pu} & \hat{\mathbf{G}}_{pp} & -\hat{\mathbf{G}}_{p\sigma} \\ \hat{\mathbf{G}}_{\sigma u} & -\hat{\mathbf{G}}_{\sigma p} & \hat{\mathbf{G}}_{\sigma\sigma} \end{bmatrix} \begin{Bmatrix} \mathbf{P} \\ \mathbf{U} \\ \mathbf{\Theta} \end{Bmatrix} - \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{\Sigma} \end{Bmatrix}\right] = 0. \tag{22}$$

Let us partition the vectors of boundary variables into subvectors pertaining to Γ_p and Γ_u, marked by subscript p and u, respectively:

$$\mathbf{P} = \begin{Bmatrix} \mathbf{P}_p \\ \mathbf{P}_u \end{Bmatrix},\quad \mathbf{U} = \begin{Bmatrix} \mathbf{U}_p \\ \mathbf{U}_u \end{Bmatrix};\quad \mathbf{F}^{**} = \begin{Bmatrix} \mathbf{F}_p^{**} \\ \mathbf{F}_u^{**} \end{Bmatrix},\quad \mathbf{D}^{**} = \begin{Bmatrix} \mathbf{D}_p^{**} \\ \mathbf{D}_u^{**} \end{Bmatrix}. \tag{23}$$

The boundary unknowns $\mathbf{P}_u$ and $\mathbf{U}_p$ will be gathered in the vector $\mathbf{X}$. A system of as many independent linear algebraic equations as boundary unknowns is generated by requiring the Betti equation (22) to hold for every $\mathbf{F}_u^{**}$, $\mathbf{D}_p^{**}$. This system can be wrriten as:

$$\mathbf{A}\mathbf{x}+\mathbf{C}\boldsymbol{\Theta}+\bar{\mathbf{B}} = \mathbf{0}, \text{ setting } \mathbf{A} \equiv \begin{bmatrix} \hat{\mathbf{G}}^{uu}_{uu} & -\hat{\mathbf{G}}^{up}_{up} \\ -\hat{\mathbf{G}}^{pu}_{pu} & \hat{\mathbf{G}}^{pp}_{pp} \end{bmatrix}, \quad \mathbf{C} \equiv \begin{bmatrix} \hat{\mathbf{G}}^{u}_{u\sigma} \\ -\hat{\mathbf{G}}^{p}_{p\sigma} \end{bmatrix} \tag{24}$$

where superscripts u and p specify submatrices defined by the vector partitions (23) in the matrices which show up in eqn (22).

In eqn (24) $\bar{\mathbf{B}}$ denotes the resulting vector which contains the boundary data $\mathbf{P}_p$ and $\mathbf{U}_u$. The coefficient matrix $\mathbf{A}$ is readily seen to be *symmetric* as a consequence of the reciprocity relations (21) and of the modelling restriction (iii). In fact, this easily fulfilled assumption makes the integral in eqn (21) vanish so that $\hat{\mathbf{G}}_{up} = \hat{\mathbf{G}}^{t}_{pu}$.

Alternatively, *non-symmetric* systems of independent equations might be generated from eqn (11). For example, using static discontinuities only ($\mathbf{F}^{**}$) on the whole boundary Γ, one arrives at the customary Somigliana equation enforced on Γ in a Galerkin weighted-residual sense (rather than by collocation) as in Parreira and Guiggiani (1989).

The arbitrariness of vector $\boldsymbol{\Theta}^{**}$ in eqn (22) gives rise to the following equation ($\mathbf{B}'$ being a new vector of data containing the external actions):

$$\mathbf{C}^{t}\mathbf{X}+\mathbf{B}'+\hat{\mathbf{G}}_{\sigma\sigma}\boldsymbol{\Theta} = \boldsymbol{\Sigma}. \tag{25}$$

Clearly, if $\boldsymbol{\Theta}$ governs a field of **assigned** initial strains (such as thermal strains), eqn (25) is decoupled from eqn (24) and yields the generalized stresses $\boldsymbol{\Sigma}$ as soon as eqn (24) is solved with respect to the boundary unknowns $\mathbf{X}$. Here, however, $\boldsymbol{\Theta}$ is interpreted as governing the unknown plastic strain field (initial strains ϑ_{ij}, not considered here for simplicity, would merely contribute to the data vectors). Hence, eqns (24) and (25), are coupled and must be combined with the plastic constitutive relations expressed in a suitable form to be discussed in the subsequent section.

PLASTIC CONSTITUTION IN GENERALIZED (CELL) STRAIN AND STRESS INCREMENTS

In this section we derive from the above material model a set of relations between generalized strains $\boldsymbol{\Theta}$ introduced through the discretized eqn (17c) and generalized stresses $\boldsymbol{\Sigma}$ defined by eqn (19). The notions and use of generalized variables and "consistent" modelling had been proposed for BE elastoplasticity by Maier (1983) and Maier and Nappi (1984) and are studied in some detail by Comi *et al.* (1991b) with reference to finite elements; hence, a concise presentation will be given here. Related works in the finite elements context by Oden and Brauchli (1971) and Corradi (1978) seem especially worth quoting.

The requirement for $\boldsymbol{\Sigma}$ and $\boldsymbol{\Theta}$ to be generalized variables in Prager's sense reads:

$$\int_{\Omega} \boldsymbol{\sigma}^{t}\boldsymbol{\varepsilon}^{p}\,\mathrm{d}\Omega = \boldsymbol{\Sigma}^{t}\boldsymbol{\Theta}, \quad \text{for any } \boldsymbol{\Sigma}, \boldsymbol{\Theta}. \tag{26}$$

Having chosen the strain shape matrix $\boldsymbol{\Psi}_{\theta}$, eqn (17c), this requirement is easily seen to be satisfied if the stresses are modelled as:

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\Psi}_{\sigma}(\mathbf{x})\boldsymbol{\Sigma}, \quad \text{where}: \boldsymbol{\Psi}_{\sigma}(\mathbf{x}) = \boldsymbol{\Psi}_{\theta}(\mathbf{x})\left[\int_{\Omega} \boldsymbol{\Psi}^{t}_{\theta}\boldsymbol{\Psi}_{\theta}\,\mathrm{d}\Omega\right]^{-1}. \tag{27}$$

Equation (27a) can be regarded as the inverse of eqn (19) which defines the generalized stress vector $\boldsymbol{\Sigma}$ as weighted averages of local stresses $\boldsymbol{\sigma}$.

Similarly, generalized yield functions $\boldsymbol{\Phi}$ and plastic multipliers $\Delta\boldsymbol{\Lambda}$ can be introduced to govern the fields of the relevant local quantities:

$$\Delta\lambda(\mathbf{x}) = \boldsymbol{\Psi}_\lambda(\mathbf{x})\Delta\boldsymbol{\Lambda}; \quad \phi(\mathbf{x}) = \boldsymbol{\Psi}_\phi(\mathbf{x})\boldsymbol{\Phi} \tag{28}$$

and will be required to satisfy the condition: $\int_\Omega \phi\Delta\lambda \, d\Omega = \boldsymbol{\Phi}^t\Delta\boldsymbol{\Lambda}$ for any $\boldsymbol{\Phi}$ and $\Delta\boldsymbol{\Lambda}$. This leads to the inverse relations of models (28) and to a dependence between their interpolation matrices:

$$\int_\Omega \boldsymbol{\Psi}_\phi^t\Delta\lambda \, d\Omega = \Delta\boldsymbol{\Lambda}, \quad \int_\Omega \boldsymbol{\Psi}_\lambda^t\phi \, d\Omega = \boldsymbol{\Phi}, \quad \boldsymbol{\Psi}_\phi = \boldsymbol{\Psi}_\lambda\left[\int_\Omega \boldsymbol{\Psi}_\lambda^t\boldsymbol{\Psi}_\lambda \, d\Omega\right]^{-1}. \tag{29}$$

Finally, generalized kinematic and static internal variables gathered in vectors **H** and **Q**, respectively, will be adopted in order to discretize by interpolations the relevant fields (**H** is a Greek letter, capital counterpart to $\boldsymbol{\eta}$):

$$\boldsymbol{\eta}(\mathbf{x}) = \boldsymbol{\Psi}_\eta(\mathbf{x})\mathbf{H}; \quad \mathbf{q}(\mathbf{x}) = \boldsymbol{\Psi}_q(\mathbf{x})\mathbf{Q}. \tag{30}$$

Once again we require the conservation of the dot product (which intervenes in the thermodynamic postulate on dissipation, eqn (7)), namely that: $\int_\Omega \mathbf{q}^t\boldsymbol{\eta} \, d\Omega = \mathbf{Q}^t\mathbf{H}$ for any **Q** and **H**. This leads to a relation set similar to (29):

$$\int_\Omega \boldsymbol{\Psi}_q^t\boldsymbol{\eta} \, d\Omega = \mathbf{H}, \quad \int_\Omega \boldsymbol{\Psi}_\eta^t\mathbf{q} \, d\Omega = \mathbf{Q}, \quad \boldsymbol{\Psi}_q = \boldsymbol{\Psi}_\eta\left[\int_\Omega \boldsymbol{\Psi}_\eta^t\boldsymbol{\Psi}_\eta \, d\Omega\right]^{-1}. \tag{31}$$

By combining the material model written in step-holonomic backward-difference form, eqns (8)–(10), with the modelling relations (17c), (19) and (27)–(31), we obtain the following plastic relationships in generalized variables for the finite-step problem defined over the time interval Δt starting from a known situation at $\bar{t} = t_n$ (barred quantities):

$$\boldsymbol{\Phi} = \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{\Sigma}^t}(\boldsymbol{\Sigma},\mathbf{Q})\boldsymbol{\Sigma} + \frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{Q}^t}(\boldsymbol{\Sigma},\mathbf{Q})\mathbf{Q} - \mathbf{Y} \leq \mathbf{0}, \quad \Delta\boldsymbol{\Lambda} \geq \mathbf{0}, \quad \boldsymbol{\Phi}^t\Delta\boldsymbol{\Lambda} = 0 \tag{32}$$

$$\Delta\boldsymbol{\Theta} = \frac{\partial\boldsymbol{\Phi}^t}{\partial\boldsymbol{\Sigma}}(\boldsymbol{\Sigma},\mathbf{Q})\Delta\boldsymbol{\Lambda}, \quad \Delta\mathbf{H} = -\frac{\partial\boldsymbol{\Phi}^t}{\partial\mathbf{Q}}(\boldsymbol{\Sigma},\mathbf{Q})\Delta\boldsymbol{\Lambda} \tag{33}$$

$$\mathbf{Q} = \frac{\partial W}{\partial\mathbf{H}}(\mathbf{H}) \tag{34}$$

having set:

$$\boldsymbol{\Sigma} = \bar{\boldsymbol{\Sigma}} + \Delta\boldsymbol{\Sigma}, \quad \mathbf{Q} \equiv \bar{\mathbf{Q}} + \Delta\mathbf{Q}, \quad \mathbf{Y} \equiv \int_\Omega \boldsymbol{\Psi}_\lambda^t y \, d\Omega, \quad W \equiv \int_\Omega w(\boldsymbol{\Psi}_\eta^t\mathbf{H}) \, d\Omega. \tag{35}$$

Equations (32)–(34) enforce the material plastic constitution in a weak, weighted-average sense and can be regarded as non-local constitutive laws written for each one of all cells simultaneously.

Similarly, elastic laws for cells can be derived from Hooke's material law (3) through eqn (19) and interpolation (17c), the latter being attributed to total strains $\boldsymbol{\varepsilon}(\mathbf{x})$ and to the relevant generalized variables gathered in vector **E**:

$$\boldsymbol{\Sigma} = \int_\Omega \boldsymbol{\Psi}_\theta^t\mathbf{k}\cdot(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \, d\Omega = \mathbf{K}\cdot(\mathbf{E} - \boldsymbol{\Theta}), \quad \text{with } \mathbf{K} \equiv \int_\Omega \boldsymbol{\Psi}_\theta^t\mathbf{k}\boldsymbol{\Psi}_\theta \, d\Omega \tag{36}$$

where **k** denotes the stiffness matrix of elastic moduli, i.e. the inverse of the elastic compliance tensor C in eqn (3).

In a parallel paper by Comi *et al.* (1990) various options for choosing the generalized variables and the relevant interpolations, are comparatively discussed from the computational standpoint. For the practically advantageous options which give rise to a decoupling (cell by cell) of the constitutive formulation (32)–(36), it is proved in the above quoted paper that the essential constitutive features carry over from the local material level to the average cell level. In particular this occurs for the features specified in the section on constitutive restrictions (normality, convexity, stability, order-one homogeneity). For example, consider the convexity assumed for the yield function expressed by the inequality (denoting by primed and unprimed symbols any two pairs of material variable vectors, $\boldsymbol{\sigma}'\mathbf{q}'$ and $\boldsymbol{\sigma}\mathbf{q}$):

$$\phi(\boldsymbol{\sigma},\mathbf{q}) \geqslant \phi(\boldsymbol{\sigma}',\mathbf{q}') + \frac{\partial \phi}{\partial \boldsymbol{\sigma}^{\mathrm{t}}}(\boldsymbol{\sigma}',\mathbf{q}')\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}') + \frac{\partial \phi}{\partial \mathbf{q}^{\mathrm{t}}}(\boldsymbol{\sigma}',\mathbf{q}')\cdot(\mathbf{q}-\mathbf{q}'). \tag{37a}$$

Assume now (as in the example to be presented later) a piecewise-constant modelling of all variables over Ω; namely, identify the generalized kinematic variables of each (say triangular or tetrahedral) cell with their local counterparts in its centroid $\bar{x}_c$, index c running over the cell set; thus, through eqns (19), (26b) and (31b): $\boldsymbol{\Sigma}^{\mathrm{t}} = \{..\Omega_c\boldsymbol{\sigma}^{\mathrm{t}}(\bar{\mathrm{x}}_c)..\}$, $\boldsymbol{\Phi}^{\mathrm{t}} = \{..\Omega_c\phi(\bar{\mathrm{x}}_c)..\}$, $\mathbf{Q}^{\mathrm{t}} = \{..\Omega_c\mathbf{q}^{\mathrm{t}}(\bar{\mathrm{x}}_c)..\}$. It is self-evident in this special case (but could be shown for other models of practical interest) that the convexity inequality (37a) gives rise to an analogous inequality in generalized variables, which means convexity of all generalized yield functions:

$$\boldsymbol{\Phi}(\boldsymbol{\Sigma},\mathbf{Q}) \geqslant \boldsymbol{\Phi}(\boldsymbol{\Sigma}',\mathbf{Q}') + \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{\Sigma}^{\mathrm{t}}}(\boldsymbol{\Sigma}',\mathbf{Q}')\cdot(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}') + \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{Q}^{\mathrm{t}}}(\boldsymbol{\Sigma}',\mathbf{Q}')\cdot(\mathbf{Q}-\mathbf{Q}'). \tag{37b}$$

Similarly, convexity of material potential $w(\boldsymbol{\eta})$ can be seen to entail convexity of the generalized potential $W(\mathbf{H})$:

$$W(\mathbf{H}) \geqslant W(\mathbf{H}') + \frac{\partial W}{\partial \mathbf{H}^{\mathrm{t}}}(\mathbf{H}')\cdot(\mathbf{H}-\mathbf{H}'). \tag{38}$$

Note that, while the material model was attributed a single yield mode (smooth yield surface in $\boldsymbol{\sigma}$ space), the cell constitutive law may exhibit a multiplicity of modes (yield surface with "corners" in $\boldsymbol{\Sigma}$ space) when the model of the yield function ϕ over cells is no longer piecewise constant.

THE DISCRETIZED ELASTIC-PLASTIC PROBLEM IN FINITE INCREMENTS

After the preceding time and space discretizations, let us now assemble the complete set of relationships which govern the response (holonomic in the step) to finite increments of external actions over the time step Δt, starting from a known state at $\bar{t} = t_n$.

Equations (32)–(36) describe the constitutive law, approximated by a backward-difference formulation and enforced in a weighted-average sense over cells.

The discretized boundary integral eqns (24) can be rewritten here for increments:

$$\mathbf{A}\Delta\mathbf{X} + \mathbf{C}\Delta\boldsymbol{\Theta} + \Delta\bar{\mathbf{B}} = \mathbf{0}. \tag{39}$$

The discretized integral eqn (25) resulting from the discretized Betti's integral eqn (22), when it is imposed to hold for any arbitrary vector $\boldsymbol{\Theta}^{**}$, provides another linear equation for increments; namely, the following representation of the generalized stress increment vector:

$$\Delta\Sigma = C^t\Delta X + \hat{G}_{\sigma\sigma}\Delta\Theta + \Delta B'. \tag{40}$$

Equations (32)–(40) define the finite-step elastic–plastic problem we are primarily concerned with in this paper.

An alternative, more compact formulation is achieved by solving once for all eqn (39) with respect to ΔX and substituting this into eqn (40). Thus we can associate with eqns (32)–(36) the single equation:

$$\Delta\Sigma^e + Z\Delta\Theta = \Delta\Sigma \tag{41}$$

having set:

$$\Delta\Sigma^e \equiv \Delta B' - C^tA^{-1}\Delta\bar{B}, \quad Z \equiv \hat{G}_{\sigma\sigma} - C^tA^{-1}C. \tag{42}$$

Vector $\Delta\Sigma^e$ defines the linear elastic stress response to the load increments. Matrix Z transforms generalized plastic strain increments into consequent stresses $\Delta\Sigma^s = \Delta\Sigma - \Delta\Sigma^e$ (self-equilibrated in the approximate sense consistent with the modelling). Matrix Z is easily seen to be negative semi-definite: in fact, $-\Delta\Theta^t\Delta\Sigma^s$ represents the strain energy stored in the body due to strains $\Delta\Theta$ conceived as external actions. Finally, Z exhibits a self-evident symmetry which would not be exhibited by its counterpart in other BE methods (A would not be symmetric in eqn (42b)).

EXTREMUM THEOREMS FOR THE DISCRETIZED ELASTIC–PLASTIC FINITE STEP PROBLEM

The solution of the elastic–plastic problem in finite increments formulated by eqns (32)–(36) and (41) after the time and BE-space discretizations carried out in what precedes, can be related to the solution of suitable minimization problems by virtue of the solution properties stated and proved in what follows.

Proposition 1. *The (any) solution to the finite-step BE-discretized problem governed by eqns* (32)–(36) *and* (41) *also solves the following optimization problem*:

$$\min\{\omega(\Delta\Theta, \Delta\Lambda, \Delta E, \Delta H) \equiv -\tfrac{1}{2}\Delta\Theta^tZ\Delta\Theta + Y^t\Delta\Lambda - (\bar{\Sigma} + \Delta\Sigma^e)^t\Delta\Theta + W(\bar{H} + \Delta H)\} \tag{43}$$

subject to the constraints:

$$\Delta\Lambda \geqslant 0, \quad \Delta\Theta = \frac{\partial\Phi^t}{\partial\Sigma}(\Sigma^*, Q)\Delta\Lambda, \quad \Delta H = -\frac{\partial\Phi^t}{\partial Q}(\Sigma^*, Q)\Delta\Lambda \tag{44}$$

where:

$$\Sigma^* \equiv \bar{\Sigma} + K\cdot(\Delta E - \Delta\Theta); \quad Q = \bar{Q} + \Delta Q = \frac{\partial W}{\partial H}(\bar{H} + \Delta H). \tag{45}$$

Proof. We will denote below by capped symbols quantities pertaining to the solution of the discretized b.v. problem in finite steps (32)–(36) and (41) and by primed symbols quantities which comply with constraints (44), i.e. "feasible" vectors for the constrained optimization (43)–(44). With these symbols one can easily realize that the following chain of relations holds:

$$\begin{aligned}\omega' - \hat{\omega} = & -\tfrac{1}{2}(\Delta\Theta' - \Delta\hat{\Theta})^tZ\cdot(\Delta\Theta' - \Delta\hat{\Theta}) + \Delta\hat{\Theta}^tZ\cdot(\Delta\hat{\Theta} - \Delta\Theta') \\ & + Y^t\cdot(\Delta\Lambda' - \Delta\hat{\Lambda}) - (\bar{\Sigma} + \Delta\Sigma^e)^t\cdot(\Delta\Theta' - \Delta\hat{\Theta}) + W(H') - W(\hat{H}) \\ & \geqslant \hat{\Sigma}^t\cdot(\Delta\hat{\Theta} - \Delta\Theta') - Y^t\cdot(\Delta\hat{\Lambda} - \Delta\Lambda') + Q^t(\hat{H})\cdot(H' - \hat{H})\end{aligned}$$

$$= \left[\frac{\partial \Phi}{\partial \Sigma^t}(\hat{\Sigma}, \hat{Q}) \cdot \hat{\Sigma} + \frac{\partial \Phi}{\partial Q^t}(\hat{\Sigma}, \hat{Q}) \cdot \hat{Q} - Y\right]^t \Delta\hat{\Lambda}$$

$$- \left[\frac{\partial \Phi}{\partial \Sigma^t}(\Sigma^{*\prime}, Q^{\prime}) \cdot \hat{\Sigma} + \frac{\partial \Phi}{\partial Q^t}(\Sigma^{*\prime}, Q^{\prime}) \cdot \hat{Q} - Y\right]^t \Delta\Lambda^{\prime} \geqslant \hat{\Phi}^t \Delta\hat{\Lambda} - \hat{\Phi}^t \Delta\Lambda^{\prime}. \quad (46)$$

In fact, noting that in eqn (46) it has been set:

$$\hat{\Sigma} = \bar{\Sigma} + \Delta\Sigma^e + Z\Delta\hat{\Theta}, \quad (47)$$

the inequalities in (46) can be justified taking into account: the symmetric negative semi-definite nature of matrix **Z** generated by the Galerkin symmetric BE formulation; the homogeneity of effective stresses; the constitutive convexity assumptions (b) and (c).

Since on the right hand side of the latter inequality (46) the first addend vanishes and in the second $\hat{\Phi} \leqslant 0$ and $\Delta\Lambda^{\prime} \geqslant 0$, the stated circumstance that $\omega^{\prime} \geqslant \hat{\omega}$ is ascertained (q.e.d.).

Note that the converse of Proposition 1 is not proved yet, i.e. an optimal vector for the optimization problem (43)–(44) was not shown to solve the finite-step problem (32)–(40). A sufficient (generally not necessary) condition for such a solution is provided by the following statement concerning another minimization problem.

Proposition 2. The set of solutions for the finite-increment discretized b.v. problem (32)–(40) *coincides with the set of Kuhn–Tucker points of the following optimization problem*:

$$\min \zeta(\Delta\Theta, \Delta\Lambda, \Delta\mathrm{K}, \Delta\mathrm{H}) \quad (48)$$

subject to:

$$\Delta\Lambda \geqslant 0 \quad (49)$$

where, using again eqns (45), *the objective function reads*:

$$\zeta \equiv -\tfrac{1}{2}\Delta\Theta^t Z \Delta\Theta + Y^t \Delta\Lambda - (\bar{\Sigma} + \Delta\Sigma^e)^t \Delta\Theta + W(\bar{H} + \Delta H)$$

$$+ \Sigma^{*t} \cdot \left[\Delta\Theta - \frac{\partial \Phi^t}{\partial \Sigma}(\Sigma^*, Q)\Delta\Lambda\right] - Q^t(\bar{H} + \Delta H) \cdot \left[\Delta H + \frac{\partial \Phi^t}{\partial Q}(\Sigma^*, Q)\Delta\Lambda\right]. \quad (50)$$

As a consequence, an optimal vector for problem (48)–(49) *also solves the finite-step problem* (32)–(40).

Proof. Let us write the Kuhn–Tucker conditions (of stationarity in the generalized sense) for the non-linear mathematical programming problem (48)–(49). Denoting by μ a vector of Lagrangian multipliers for constraints (49) and adopting index α over the set of generalized yield functions and the summation convention for α, the Kuhn–Tucker conditions read:

$$-Z\Delta\Theta - \bar{\Sigma} - \Delta\Sigma^e - K \cdot \left[\Delta\Theta - \frac{\partial \Phi^t}{\partial \Sigma}(\Sigma^*, Q)\Delta\Lambda\right]$$

$$+ \left[I + K\frac{\partial^2 \Phi_\alpha}{\partial \Sigma\, \partial \Sigma^t}\Delta\Lambda_\alpha\right]\Sigma^* + \left[K\frac{\partial^2 \Phi_\alpha}{\partial \Sigma\, \partial Q^t}\Delta\Lambda_\alpha\right]Q = 0 \quad (51)$$

$$Q + \left[-\frac{\partial Q}{\partial H^t}\frac{\partial^2 \Phi_\alpha}{\partial Q\, \partial \Sigma^t}\Delta\Lambda_\alpha\right]\Sigma^* + \frac{\partial Q^t}{\partial H} \cdot \left[\Delta H + \frac{\partial \Phi^t}{\partial Q}(\Sigma^*, Q)\Delta\Lambda\right]$$

$$-\left[I + \frac{\partial Q}{\partial H^t}\frac{\partial^2 \Phi_\alpha}{\partial Q\, \partial Q^t}\Delta\Lambda_\alpha\right]Q = 0 \quad (52)$$

$$\mathbf{K}\cdot\left[\Delta\Theta-\frac{\partial\Phi^{t}}{\partial\Sigma}(\Sigma^{*},\mathbf{Q})\Delta\Lambda\right]-\mathbf{K}\cdot\left[\frac{\partial^{2}\Phi_{\alpha}}{\partial\Sigma\,\partial\Sigma^{t}}\Sigma^{*}+\frac{\partial^{2}\Phi_{\alpha}}{\partial\Sigma\,\partial\mathbf{Q}^{t}}\mathbf{Q}\right]\Delta\Lambda_{\alpha}=\mathbf{0} \tag{53}$$

$$\mathbf{Y}-\frac{\partial\Phi}{\partial\Sigma^{t}}(\Sigma^{*},\mathbf{Q})\Sigma^{*}-\frac{\partial\Phi}{\partial\mathbf{Q}^{t}}(\Sigma^{*},\mathbf{Q})\mathbf{Q}=\mu \tag{54}$$

$$\Delta\Lambda\geqslant\mathbf{0},\quad \mu\geqslant\mathbf{0},\quad \mu^{t}\Delta\Lambda=0. \tag{55}$$

Substituting eqn (53) into (51), we obtain an equation which expresses the equilibrium of stresses Σ^* defined by eqn (45a):

$$\Sigma^{*}=\hat{\Sigma}+\Delta\Sigma^{e}+\mathbf{Z}\Delta\Theta. \tag{56}$$

In view of the positive first-order homogeneity hypothesis (d) and as a consequence of Euler's theorem on homogeneous functions, one can easily realize that the following property of the Hessian of Φ_α holds true:

$$\left[\begin{Bmatrix}\dfrac{\partial}{\partial\Sigma}\\ \dfrac{\partial}{\partial\mathbf{Q}}\end{Bmatrix}\begin{Bmatrix}\dfrac{\partial\Phi_{\alpha}}{\partial\Sigma^{t}} & \dfrac{\partial\Phi_{\alpha}}{\partial\mathbf{Q}^{t}}\end{Bmatrix}\right]_{\Sigma,\mathbf{Q}}\cdot\begin{Bmatrix}\Sigma\\ \mathbf{Q}\end{Bmatrix}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\end{Bmatrix}. \tag{57}$$

Therefore, taking account of (56), eqn (53) reduces to:

$$\Delta\Theta=\frac{\partial\Phi^{t}}{\partial\Sigma}(\Sigma^{*},\mathbf{Q})\Delta\Lambda \tag{58}$$

and, if $\partial\mathbf{Q}^t/\partial\mathbf{H}$ is non-singular (i.e. in view of hypothesis (c), W is strictly convex), eqn (51) yields:

$$\Delta\mathbf{H}=-\frac{\partial\Phi^{t}}{\partial\mathbf{Q}}(\Sigma^{*},\mathbf{Q})\Delta\Lambda. \tag{59}$$

Now, by associating eqns (56), (58) and (59) to eqns (54) and (55), one recovers the complete set of relations governing the finite-step problem and, thus, shows its equivalence with the Kuhn–Tucker conditions of problem (50) (q.e.d.).

It is worth noting that the Kuhn–Tucker conditions are necessary for optimality, but generally not sufficient, unless the optimization problem is convex. Problem (48)–(50) is not convex in general and, hence, Proposition 2 does not imply its equivalence to the finite-step problem. However, on the basis of the preceding Proposition 2, we can now prove the converse of Proposition 1, i.e. the following statement:

Proposition 3. *The (any) solution (optimal vector) to the minimization problem* (43)–(45) *solves the finite-increment discretized b.v. problem* (32)–(36) *and* (41). *This circumstance and that stated by Proposition* 1 *make the two problems equivalent.*

Proof. We denote by $\mathbb{M}_\omega$, $\mathbb{K}_\omega$ and $\mathbb{A}_\omega$ the sets of optimal vectors, Kuhn–Tucker points and admissible (or feasible) vectors, respectively, for the non-linear programming problem (43)–(45) with objective function ω. Symbols $\mathbb{M}_\zeta$, $\mathbb{K}_\zeta$ and $\mathbb{A}_\zeta$ will indicate the corresponding sets for the problem (48)–(50) with objective function ζ.

Due to their very meanings, these sets satisfy the relations:

$$\mathbb{M}_{\omega} \subseteq \mathbb{K}_{\omega} \subseteq \mathbb{A}_{\omega}; \quad \mathbb{M}_{\zeta} \subseteq \mathbb{K}_{\zeta} \subseteq \mathbb{A}_{\zeta}. \tag{60}$$

Let $\mathbb{S}$ denote the set of solutions to the finite-increment problem (32)–(40). Propositions 1 and 2 state, respectively, that:

$$\mathbb{S} \subseteq \mathbb{M}_{\omega}; \quad \mathbb{S} = \mathbb{K}_{\zeta}. \tag{61}$$

Using eqns (60) and (61) we may write:

$$\mathbb{M}_{\zeta} \subseteq \mathbb{S} \subseteq \mathbb{M}_{\omega} \subseteq \mathbb{A}_{\omega}. \tag{62}$$

By inspecting comparatively the constraints of the two (ω and ζ) optimization problems, we notice that:

$$\mathbb{A}_{\omega} \subset \mathbb{A}_{\zeta}, \quad \omega(\mathbb{A}_{\omega}) = \zeta(\mathbb{A}_{\omega}). \tag{63}$$

The latter relation means that the two objective functions coincide over the feasible domain $\mathbb{A}_{\omega}$ of problem ω. Equations (63) imply that $\mathbb{M}_{\zeta} = \mathbb{M}_{\omega}$ and, hence, through eqn (62), that $\mathbb{S} = \mathbb{M}_{\omega}$. Thus the stated equivalence is justified and, therefore (through Proposition 1), a proof is also reached for the stated sufficient condition for the b.v. problem solution (q.e.d.).

It is worth noting that the above conclusions represent *ad hoc* statements for the present BE-formulation and are distinct from, though related to, those established for continua by Comi *et al.* (1991a), and quite distinct from those derived by convex analysis notions e.g. by Martin *et al.* (1987). They turn out computationally fruitful in as much as they provide a basis for the convergence criteria discussed in the next section by a path of reasoning in a sense parallel to the one followed by Comi and Maier (1990) in a finite element context.

AN ALGORITHM AND A CONVERGENCE THEOREM FOR THE ITERATIVE SOLUTION OF THE STEP PROBLEM

The finite-step problem (32)–(36) and (41) can be numerically solved by the following iterative algorithm ("successive substitutions" or modified Newton–Raphson algorithm). This is basically the conventional pattern adopted in the literature on non-symmetric BEM in plasticity, as e.g. in Brebbia *et al.* (1984) and Cruse and Polch (1986).

(1) Generate the coefficient matrices once for all: $\mathbf{A}$, $\mathbf{C}$, $\hat{\mathbf{G}}_{\sigma\sigma}$ and, consequently, via eqn (42b), matrix $\mathbf{Z}$.

(2) For the current step compute the linear-elastic stress response $\Delta\mathbf{\Sigma}^e$ to the given load increments, eqn (42a).

(3) Initialization: for $i = 1$ assume either $\Delta\mathbf{\Theta} = \mathbf{0}$ or $\Delta\mathbf{\Theta}$ equal to the best guess based on the preceding loading step.

(4) Prediction: from the iterate $i-1$ of plastic strain increments, compute the new stress increments, $\Delta\mathbf{\Sigma}$ through eqn (41).

(5) Correction: compute through the constitutive law (32)–(34) the new plastic strain increments $\Delta\mathbf{\Theta}$.

(6) Termination test: if the changes in $\Delta\mathbf{\Theta}$ from iteration $i-1$ to i do not exceed in a norm a preassigned tolerance, enter the data of a new loading step and perform phase 2.

Assume that the first $i-1$ iterations have led to plastic strain increments $\Delta\mathbf{\Theta}^{i-1}$. According to the envisaged algorithm, the ith iterate is generated by the following computations:

$$\Delta\mathbf{\Sigma}^{i-1} = \Delta\mathbf{\Sigma}^{e} + \mathbf{Z}\Delta\mathbf{\Theta}^{i-1}, \quad \Delta\mathbf{E}^{i} = \mathbf{K}^{-1}\Delta\mathbf{\Sigma}^{i-1} + \Delta\mathbf{\Theta}^{i-1} \tag{64}$$

$$\Delta\mathbf{\Theta}^{i} = \frac{\partial\mathbf{\Phi}^{t}}{\partial\mathbf{\Sigma}}(\mathbf{\Sigma}^{*i}, \mathbf{Q}^{i})\Delta\mathbf{\Lambda}^{i}, \quad \Delta\mathbf{H}^{i} = -\frac{\partial\mathbf{\Phi}^{t}}{\partial\mathbf{Q}}(\mathbf{\Sigma}^{*i}, \mathbf{Q}^{i})\Delta\mathbf{\Lambda}^{i}, \quad \Delta\mathbf{\Lambda}^{i} \geqslant \mathbf{0} \tag{65}$$

$$\mathbf{\Phi}^{i} = \frac{\partial\mathbf{\Phi}}{\partial\mathbf{\Sigma}^{t}}(\mathbf{\Sigma}^{*i}, \mathbf{Q}^{i})\mathbf{\Sigma}^{*i} + \frac{\partial\mathbf{\Phi}}{\partial\mathbf{Q}^{t}}(\mathbf{\Sigma}^{*i}, \mathbf{Q}^{i})\mathbf{Q}^{i} - \mathbf{Y} \leqslant \mathbf{0}, \quad \mathbf{\Phi}^{i t}\Delta\mathbf{\Lambda}^{i} = 0 \tag{66}$$

$$\mathbf{\Sigma}^{*i} \equiv \bar{\mathbf{\Sigma}} + \mathbf{K}\cdot(\Delta\mathbf{E}^{i} - \Delta\mathbf{\Theta}^{i}); \quad \mathbf{Q}^{i} = \frac{\partial W}{\partial\mathbf{H}}(\bar{\mathbf{H}} + \Delta\mathbf{H}^{i}). \tag{67}$$

The linear "**prediction**" (64), resting only on the discretized integral equations, enforces equilibrium and compatibility in the system as a whole, supposed to be linear-elastic.

The "**correction**" restores the elastic–plastic constitutive law at the price of new equilibrium violations; it requires the solution of the non-linear equations and inequalities (65)–(67) in $\Delta\mathbf{\Lambda}^{i}$, $\Delta\mathbf{\Theta}^{i}$ and $\Delta\mathbf{H}^{i}$, but is carried out locally cell by cell in a decoupled form [cp. Comi *et al.* (1991b)].

The equivalence between the discretized b.v. problem in finite increments and a minimization problem established by Proposition 3, leads to the following, computationally meaningful result concerning the above solution technique.

Proposition 4. *The iterative "successive substitution" method specified by eqns* (64)–(67) *for solving the finite-step problem* (32)–(36) *and* (41) *does converge to the* (*or to a*) *solution of it.*

Proof. Let us evaluate the difference between the values ω^{i-1} and ω^{i} of the objective function ω, eqn (43), at the end of iterations $i-1$ and i respectively:

$$\begin{aligned}\Delta\omega \equiv \omega^{i-1} - \omega^{i} = {} & -\tfrac{1}{2}(\Delta\mathbf{\Theta}^{i} - \Delta\mathbf{\Theta}^{i-1})^{t}\mathbf{Z}\cdot(\Delta\mathbf{\Theta}^{i} - \Delta\mathbf{\Theta}^{i-1}) + \Delta\mathbf{\Theta}^{i t}\mathbf{Z}\cdot(\Delta\mathbf{\Theta}^{i} - \Delta\mathbf{\Theta}^{i-1}) \\ & - \mathbf{Y}^{t}\cdot(\Delta\mathbf{\Lambda}^{i} - \Delta\mathbf{\Lambda}^{i-1}) + (\bar{\mathbf{\Sigma}} + \Delta\mathbf{\Sigma}^{e})^{t}\cdot(\Delta\mathbf{\Theta}^{i} - \Delta\mathbf{\Theta}^{i-1}) + W(\bar{\mathbf{H}} + \Delta\mathbf{H}^{i-1}) - W(\bar{\mathbf{H}} + \Delta\mathbf{H}^{i}).\end{aligned} \tag{68}$$

In view of the negative semi-definiteness of matrix $\mathbf{Z}$ and of the convexity of function W and making use of eqns (67), the difference $\Delta\omega$ can be bounded from below as follows:

$$\begin{aligned}\Delta\omega \geqslant {} & \mathbf{\Sigma}^{*i t}\cdot(\Delta\mathbf{\Theta}^{i} - \Delta\mathbf{\Theta}^{i-1}) - \mathbf{Q}^{i t}\cdot(\Delta\mathbf{H}^{i} - \Delta\mathbf{H}^{i-1}) - \mathbf{Y}^{t}\cdot(\Delta\mathbf{\Lambda}^{i} - \Delta\mathbf{\Lambda}^{i-1}) \\ & + (\bar{\mathbf{\Sigma}} + \Delta\mathbf{\Sigma}^{e} + \mathbf{Z}\Delta\mathbf{\Theta}^{i} - \mathbf{\Sigma}^{*i})^{t}\cdot(\Delta\mathbf{\Theta}^{i} - \Delta\mathbf{\Theta}^{i-1}).\end{aligned} \tag{69}$$

Substituting eqns (65a,b), (67a) and (64b) into eqn (69), taking into account the homogeneity and the convexity (37b) of $\mathbf{\Phi}^{i}$ and, finally, rearranging, one obtains:

$$\Delta\omega \geqslant \mathbf{\Phi}^{i t}\Delta\mathbf{\Lambda}^{i} - \mathbf{\Phi}^{i t}\Delta\mathbf{\Lambda}^{i-1} + (\Delta\mathbf{\Theta}^{i} - \Delta\mathbf{\Theta}^{i-1})^{t}\cdot(\mathbf{Z} + \mathbf{K})\cdot(\Delta\mathbf{\Theta}^{i} - \Delta\mathbf{\Theta}^{i-1}). \tag{70}$$

The first addend is equal to zero by virtue of eqn (66c); the second is non-negative due to eqns (65c) and (66b); the last addend is also non-negative. In order to ascertain the third of these circumstances, consider the quadratic form associated with matrix $\mathbf{Z}+\mathbf{K}$ and give it alternative expressions as follows:

$$\begin{aligned}\mathbf{\Theta}^{t}\cdot(\mathbf{Z}+\mathbf{K})\mathbf{\Theta} = -\mathbf{\Theta}^{t}\mathbf{Z}\mathbf{K}^{-1}\mathbf{Z}\mathbf{\Theta} + \mathbf{\Theta}^{t}\mathbf{K}\mathbf{\Theta} = {} & (\mathbf{\Theta} + \mathbf{K}^{-1}\mathbf{Z}\mathbf{\Theta})^{t}\mathbf{K}\cdot(\mathbf{\Theta} + \mathbf{K}^{-1}\mathbf{Z}\mathbf{\Theta}) \\ & - 2\mathbf{\Theta}^{t}\mathbf{Z}\cdot(\mathbf{\Theta} + \mathbf{K}^{-1}\mathbf{Z}\mathbf{\Theta}) \geqslant 0.\end{aligned} \tag{71}$$

The latter equality in eqn (71) is readily seen to be an algebraic identity; the former is a consequence of the virtual work equation which also makes the last addend in eqn (71) vanish (because $\mathbf{Z\Theta}$ represents self-equilibrated stresses and the term in parenthesis compatible strains). Thus, the positive definiteness of $\mathbf{K}$ makes the final inequality hold.

Alternatively, this inequality can be derived from the stronger statement:

$$\tfrac{1}{2}\mathbf{\Theta}^t\mathbf{K}\mathbf{\Theta} > -\tfrac{1}{2}\mathbf{\Theta}^t\mathbf{Z}\mathbf{\Theta} \quad \text{for any } \mathbf{\Theta} \neq \mathbf{0}. \tag{72}$$

In order to justify the strict inequality (72), identify its left hand side in view of eqn (36) as the elastic strain energy due to the strain field defined by $\mathbf{\Theta}$ through the interpolations $\mathbf{\Psi}_u(\mathbf{x})$ when the displacements are set equal to zero everywhere. The right hand side in eqn (72) represents the strain energy when the displacements are forced to vanish on the constrained boundary Γ_u only. The latter situation may be transformed into the former by imposing in Ω the distribution of the reactive (body) forces supplied by the fictitious constraints which make the displacements vanish. Since these forces are not identically zero for $\mathbf{\Theta} \neq \mathbf{0}$, a positive addend of elastic energy must be added to the right hand side of eqn (72). This implies, through eqn (70) and the preceding conclusions on its addends, that $\Delta\omega \geq 0$ and that $\Delta\omega = 0$ if and only if $\Delta\mathbf{\Theta}^i = \Delta\mathbf{\Theta}^{i-1}$ (q.e.d.).

NON-LINEAR STABILITY OF THE TIME INTEGRATION PROCEDURE

In general terms, the evolution in time of a mechanical system is said to be stable if a perturbation in the initial conditions is attenuated as time ellapses. Stability in this sense has been discussed for continuum initial-boundary value problems (like eqns (1)–(7) with suitable constitutive restrictions) and for their time discretizations (like the Euler-backward scheme, eqns (8)–(10)), separately from or in combination with finite element discretizations in space. Representative contributions are due to Butcher (1975), Nguyen (1977), Simo (1991), Simo and Govindjee (1991) and Reddy and Martin (1990). Contractivity or, better, non-expansivity, B- or non-linear stability (at difference from A- or linear stability for linear stepping algorithms) represent alternative denominations used in the literature for the property in point.

The stability property of the original set of non-linear differential relations governing the behaviour of an inelastic continuum may or may not be inherited by a numerical solution method characterized by a space discretization and a time-stepping procedure. Establishing that it is inherited usually presents a non-trivial task, which has not, to the writers' knowledge, been tackled in the BE context.

The desirable stability property to assess here concerns the flow of successive finite-step solutions generated by the algorithm described in what precedes for the symmetric BE elastic–plastic analysis. Namely, an algorithm-independent norm of the difference between the original and the perturbed step-solution will be shown below to decrease or at least not to increase along the step sequence, assuming the further constitutive hypothesis of linear hardening (including perfect plasticity). More precisely, with the present symbology this is expressed by the following statement.

Proposition 5. Consider the generalized stresses and internal variables $\bar{\mathbf{\Sigma}}$, $\bar{\mathbf{Q}}$ *which define the state of the BE-discretized system at time* t_n *and vectors* $\bar{\tilde{\mathbf{\Sigma}}}$, $\bar{\tilde{\mathbf{Q}}}$ *which define a "perturbed" state of it at* t_n. *The perturbation implies* $\bar{\tilde{\mathbf{\Theta}}} \neq \bar{\mathbf{\Theta}}$ *but* $\bar{\tilde{\mathbf{\Sigma}}} = \mathbf{Z}\bar{\tilde{\mathbf{\Theta}}} + \bar{\mathbf{\Sigma}}^e$ *does not violate the (approximate) equilibrium with the load at* t_n. *Let* $\mathbf{\Sigma} = \bar{\mathbf{\Sigma}} + \Delta\mathbf{\Sigma}$, $\mathbf{Q} = \bar{\mathbf{Q}} + \Delta\mathbf{Q}$ *and* $\tilde{\mathbf{\Sigma}} = \bar{\tilde{\mathbf{\Sigma}}} + \Delta\tilde{\mathbf{\Sigma}}$, $\tilde{\mathbf{Q}} = \bar{\tilde{\mathbf{Q}}} + \Delta\tilde{\mathbf{Q}}$ *denote the two pairs of generalized variable vectors at the instant* $t_{n+1} = t_n + \Delta t$, *generated by the solution of the finite-step problem, eqns* (32)–(40), *for the given load increments (captured in the elastic stress increments* $\Delta\mathbf{\Sigma}^e$*) starting from the actual and from the perturbated state, respectively. Assume strictly stable, linearly hardening material so that eqn* (34) *becomes*:

$$\mathbf{Q} = \mathbf{M}\mathbf{H} \quad \text{where} \quad \mathbf{M} \equiv \int_{\Omega} \mathbf{\Psi}_\eta^t \mathbf{m} \mathbf{\Psi}_\eta \, d\Omega, \tag{73}$$

$\mathbf{m}$ *being the constant positive definite Hessian matrix of* $w(\boldsymbol{\eta})$. *Then, the following "non-expansivity" inequality holds*:

$$\tfrac{1}{2}(\tilde{\Sigma}-\Sigma)^t\mathbf{K}^{-1}\cdot(\tilde{\Sigma}-\Sigma)+\tfrac{1}{2}(\tilde{\mathbf{Q}}-\mathbf{Q})^t\mathbf{M}^{-1}\cdot(\tilde{\mathbf{Q}}-\mathbf{Q})$$
$$\leqslant \tfrac{1}{2}(\tilde{\bar{\Sigma}}-\bar{\Sigma})^t\mathbf{K}^{-1}\cdot(\tilde{\bar{\Sigma}}-\bar{\Sigma})+\tfrac{1}{2}(\tilde{\bar{\mathbf{Q}}}-\bar{\mathbf{Q}})^t\mathbf{M}^{-1}\cdot(\tilde{\bar{\mathbf{Q}}}-\bar{\mathbf{Q}}). \quad (74)$$

Proof. First note that the assumption of strictly stable linearly hardening material means that the material constitutive eqn (6) specializes to $q_h = m_{hk}\eta_k$ so that $w = \frac{1}{2}\eta^t\mathbf{m}\eta$ with matrix $\mathbf{m} \equiv [m_{hk}]$ positive definite. As a consequence through eqn (35d), eqn (34) in the generalized internal variables reduces to eqn (73) and the Hessian matrix of generalized potential W with respect to the generalized kinematic internal variables $\mathbf{H}$ ($\mathbf{M} \equiv \partial^2 W/\partial\mathbf{H}\,\partial\mathbf{H}^t$) turns out to be symmetric, positive definite and constant with respect to $\mathbf{H}$. Thus the convexity property, eqn (38), is satisfied *a fortiori*.

The difference d between the left hand side and the right hand side of eqn (74) can be expressed in the form:

$$d = \tfrac{1}{2}(\tilde{\Sigma}-\Sigma+\tilde{\bar{\Sigma}}-\bar{\Sigma})^t\mathbf{K}^{-1}\cdot(\Delta\tilde{\Sigma}-\Delta\Sigma)+\tfrac{1}{2}(\tilde{\mathbf{Q}}-\mathbf{Q}+\tilde{\bar{\mathbf{Q}}}-\bar{\mathbf{Q}})^t\mathbf{M}^{-1}\cdot(\Delta\tilde{\mathbf{Q}}-\Delta\mathbf{Q}). \quad (75)$$

In the two quadratic addends of eqn (75), let the first factors be split into two addends through a readily verified identity; let $\Delta\Sigma$ and $\Delta\Theta$ be rewritten using the step governing eqns (41) and (73a), respectively. Thus eqn (75) becomes:

$$d = (\tilde{\Sigma}-\Sigma)^t\mathbf{K}^{-1}\mathbf{Z}\cdot(\Delta\tilde{\Theta}-\Delta\Theta)-\tfrac{1}{2}(\Delta\tilde{\Sigma}-\Delta\Sigma)^t\mathbf{K}^{-1}\cdot(\Delta\tilde{\Sigma}-\Delta\Sigma)$$
$$+(\tilde{\mathbf{Q}}-\mathbf{Q})^t\mathbf{M}^{-1}\mathbf{M}\cdot(\Delta\tilde{\mathbf{H}}-\Delta\mathbf{H})-\tfrac{1}{2}(\Delta\tilde{\mathbf{Q}}-\Delta\mathbf{Q})^t\mathbf{M}^{-1}\cdot(\Delta\tilde{\mathbf{Q}}-\Delta\mathbf{Q}). \quad (76)$$

The second and the fourth term on the right hand side of eqn (76) are, clearly, non-positive. The first addend can be transformed using the virtual work equation:

$$(\tilde{\Sigma}-\Sigma)^t[\mathbf{K}^{-1}\mathbf{Z}\cdot(\Delta\tilde{\Theta}-\Delta\Theta)+(\Delta\tilde{\Theta}-\Delta\Theta)] = 0 \quad (77)$$

and, subsequently, the constitutive eqn (33a) expressing normality. In the third term we make use of the constitutive eqn (33b). Thus, eqn (76) gives rise to the inequality:

$$d \leqslant (\tilde{\Sigma}-\Sigma)^t\cdot\left[-\frac{\partial\Phi^t}{\partial\Sigma}(\tilde{\Sigma},\tilde{\mathbf{Q}})\Delta\tilde{\Lambda}+\frac{\partial\Phi^t}{\partial\Sigma}(\Sigma,\mathbf{Q})\Delta\Lambda\right]$$
$$+(\tilde{\mathbf{Q}}-\mathbf{Q})^t\cdot\left[-\frac{\partial\Phi^t}{\partial\mathbf{Q}}(\tilde{\Sigma},\tilde{\mathbf{Q}})\Delta\tilde{\Lambda}+\frac{\partial\Phi^t}{\partial\mathbf{Q}}(\Sigma,\mathbf{Q})\Delta\Lambda\right]. \quad (78)$$

The convexity of the generalized yield functions Φ, eqn (37b), leads to the following upper bound on the right hand side of eqn (78) and, hence, on the difference d:

$$d \leqslant [\Phi(\Sigma,\mathbf{Q})-\Phi(\tilde{\Sigma},\tilde{\mathbf{Q}})]^t\cdot(\Delta\tilde{\Lambda}-\Delta\Lambda). \quad (79)$$

Since the complementary constitutive relations (32) must be complied with both by the undisturbed and perturbed step solution, it is easily seen from eqn (79) that $d \leqslant 0$ as a consequence (q.e.d.).

Remarks. (a) Inequality (74) means that a measure of a disturbance does not increase from the onset to the end of a finite loading step, if one solves the relevant discrete non-linear step-problem generated as proposed herein (i.e. symmetric BE in space, Euler backward-difference in time). This non-linear stability property turns out to be independent from the step amplitude (unconditional stability). The perturbation measure in eqn (74) can be regarded as a "natural" energy norm [cp. Simo (1991)]. In fact, it has the mechanical meaning of complementary Helmholz "free energy" associated with the difference between the actual and the disturbed evolution.

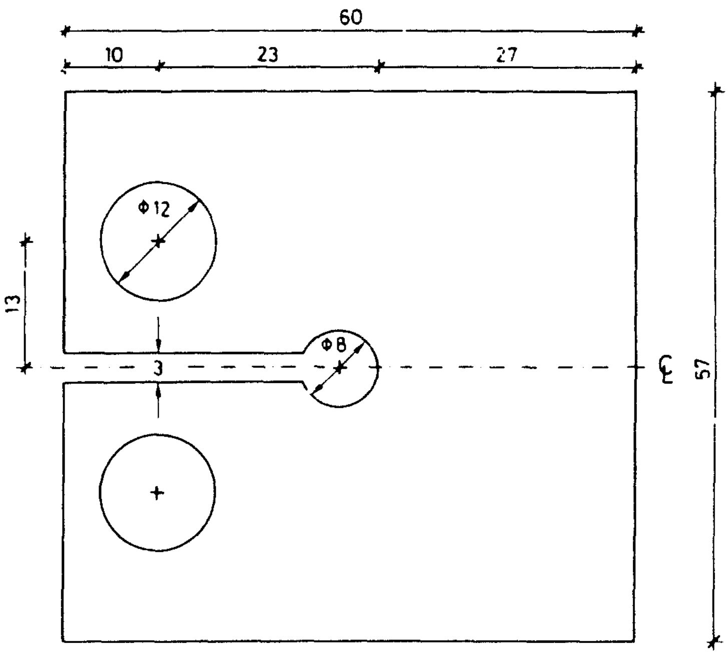

Fig. 1. A steel specimen for fracture tests (lengths in mm).

(b) The limitation of the preceding stability theorem (Proposition 5) to linear hardening was dictated by the mathematical simplicity and the brevity of the proof. Note that perfect plasticity is covered as a special case for vanishing locked-in strain energy ($w \equiv 0$, $W \equiv 0$) and, hence, for yield functions not affected by static internal variables ($\mathbf{q} \equiv \mathbf{0}$, $\mathbf{Q} \equiv \mathbf{0}$). Therefore, for perfectly plastic solids the latter quadratic form in eqn (74) is missing and the stability statement concerns a norm of stress differences only. In fact, in ideal plasticity uniqueness is guaranteed in the stress response history only; the evolution of the kinematic variables can exhibit multiple solutions which form bounded or unbounded sets, corresponding to "pseudomechanisms" or collapse mechanisms, respectively [cp. e.g. Smith and Munro (1978)].

NUMERICAL TESTS

The fracture specimen depicted in Fig. 1 and interpreted as a plane stress system, will be analyzed below for testing and illustrating the preceding theoretical results. The material is steel conceived as an elastic perfectly plastic von Mises material characterized by elastic moduli $E = 210{,}700$ MPa, $\nu = 0.27$, and by a yield stress $\sigma_y = 560$ MPa. Details on the 2D computer implementation of the symmetric Galerkin BE method adopted herein are presented elsewhere (Maier *et al.*, 1991).

Figure 2 shows the adopted subdivision of the boundary Γ into BEs and of the

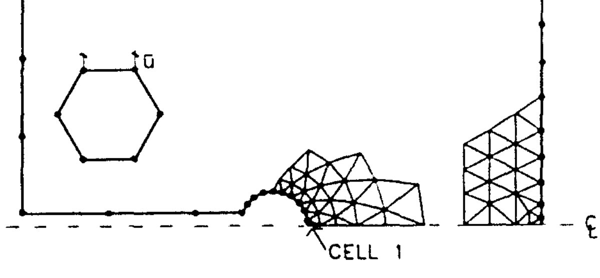

Fig. 2. Mesh of the boundary elements and cells; $\bar{u}$ is the vertical displacement, constant on the relevant element, imposed as external actions.

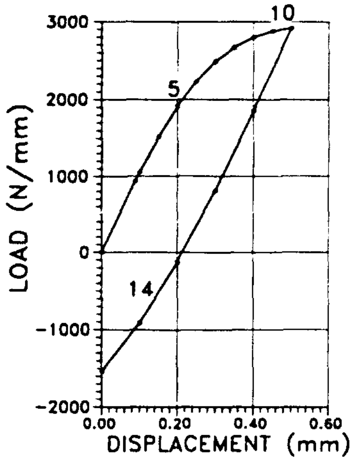

Fig. 3. A cycle of imposed displacements $\bar{u}$ analysed in 15 loading steps: computed reactive force versus $\bar{u}$.

potentially yielding subdomain Ω_p into cells. The interpolations adopted are: linear for boundary displacements; constant for boundary tractions; constant for strains and plastic multipliers over domain cells, being understood that the conjugate static variables (Σ and Φ) are "consistently" defined as generalized variables according to the path of reasoning expounded in an earlier section. The reflective symmetry of the system is imposed on the variables in the solution process, so that the axis of symmetry need not be discretized (as usual in BE analysis). A cycle $0 \div 0.5 \div 0$ mm of imposed transversal displacement $\bar{u}$ (Fig. 2) is performed and subdivided into 15 steps marked in Fig. 3. In this figure the computed resultant of the reactive tractions (per unit thickness) provided by the rigid displacement-controlling device is plotted (in N mm^{-1}) as a function of $\bar{u}$ (in millimetres). The iterative solution processes of three loading steps (Nos 5, 10, 14) were examined assuming the termination tolerance at cell level (index c):

$$\frac{\|\Delta\Theta_c^r\| - \|\Delta\Theta_c^{r-1}\|}{\|\Delta\Theta^r\|_{\max}} \leqslant 10^{-4} \tag{80}$$

where the norm $\|\cdot\|$ is defined as the von Mises equivalent plastic strain and $\|\Delta\Theta^r\|_{\max}$ is the maximum, over the cells, of the von Mises equivalent strain increment in the current iteration r.

As expected from the developed theory, Fig. 4 shows the decrease to a minimum of the energy function ω (per unit thickness, hence in N) of the equivalent minimization, eqns (43)–(45), along the iteration sequence of the three finite-step solutions by the "successive substitutions" algorithm described earlier. The abscissae of Fig. 4 are the values of a meaningful independent variable of ω, namely the non-dimensional plastic multiplier of the cell No. 1 on the symmetry axis at the bottom of the specimen notch. Along the graphs every interval between two square marks corresponds to 20 iterations of the solution procedures.

In Fig. 5 the number of these iterations is taken as the abscissae and again the objective function ω as the ordinates. The flattening of the three curves and the relatively large numbers of iterations are due to the very small tolerance chosen for the termination test. As expected the speed of convergence turns out to be smaller for higher numbers of "active" (yielding) cells (i.e. from step 14, to 5, to 10).

CONCLUSIONS

The basis of this study is provided by the symmetric Galerkin double-integration formulation of BE inelastic analysis proposed by Maier and Polizzotto (1987) and developed

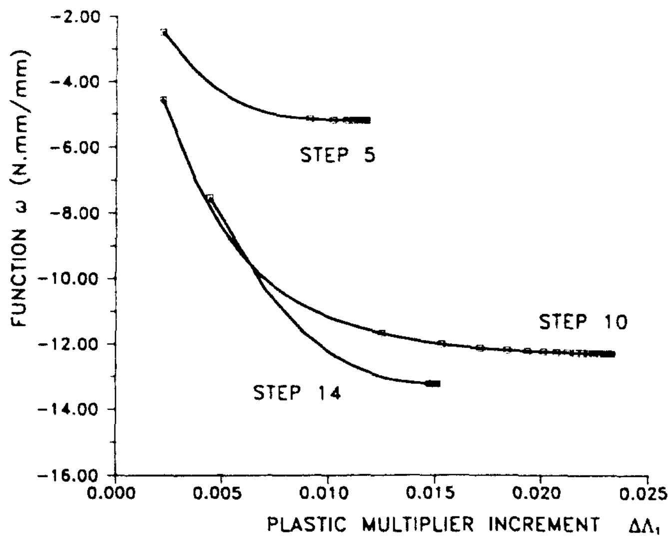

Fig. 4. Objective function Ω of the equivalent optimization versus $\Delta\Lambda_1$ of cell 1 at the bottom of the notch, for the iterative solutions of three steps (a mark every twenty iterations).

as for the computer implementation (in 2D) by Sirtori *et al.* (1991) and Maier *et al.* (1991). With reference to this method, the following results have been established in what precedes.

The symmetric BE formulation, in terms of cell generalized variables generated by suitable weighted-average enforcement of the constitutive law, has been extended to general associative elastic plastic material models with internal variables.

The finite-step stepwise-holonomic problem has been formulated by an implicit Euler backward time integration scheme and its solution was proved to be equivalent, under suitable constitutive stability conditions, to the solution of a generally non-convex constrained optimization (non-linear programming problem). As a consequence, an iterative, modified Newton Raphson method for solving the step problem has been shown to converge on the finite increment solution. Numerical tests confirmed and illustrated the extremum and convergence properties pointed out.

The approximate time integration method considered, under the restriction of linear hardening was shown to exhibit non-linear stability, in the sense that a perturbation cannot grow along the flow of step solutions.

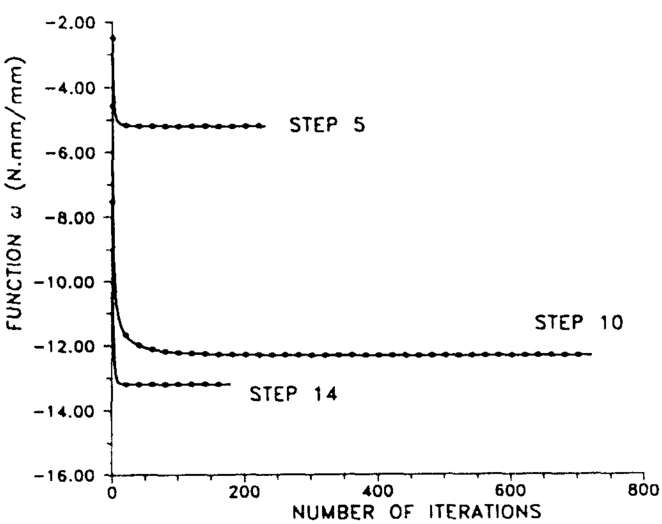

Fig. 5. Objective function Ω versus iteration counter for the iterative solutions of three loading steps.

It is believed that the conclusions of this paper cannot be reached in the framework of the traditional (non-symmetric) BE formulations and that this fact represents a remarkable advantage of the symmetric Galerkin formulation adopted herein.

Acknowledgements—The writers wish to thank A. Corigliano and S. Miccoli for helpful discussions; G.N.D.T. for financial support.

REFERENCES

Banerjee, P. K. and Butterfield, R. (1981). *Boundary Element Methods in Engineering Science*. McGraw-Hill, New York.

Brebbia, C. A., Telles, J. C. F. and Wrobel, L. C. (1984). *Boundary Element Techniques*. Springer, Berlin.

Bui, H. D. (1977). An integral equation method for solving the problem of a plane crack of arbitrary shape. *J. Mech. Phys. Solids* **25**, 29–39.

Bui, H. D. (1990). Simmétrie, dualité et variable conjuguées dans les équations intégrales de frontière. *C. R. Acad. Sci. Paris* **311**, Série II, 7–13.

Butcher, J. C. (1975). A stability property of implicit Runge–Kutta methods. *BIT* **15**, 358–361.

Caddemi, S. and Martin, J. B. (1991). Convergence of the Newton–Raphson algorithm in elastic plastic incremental analysis. *Int. J. Num. Meth. Engng* **31**, 177–191.

Comi, C., Corigliano, A. and Maier, G. (1991a). Extremum properties of finite-step solutions in elastoplasticity with nonlinear mixed hardening. *Int. J. Solids Structures* **27**(8), 965–981.

Comi, C., Maier, G. and Perego, U. (1991b). Generalized variable finite element modelling and extremum theorems in stepwise holonomic elastoplasticity with internal variables. *Comp. Meth. Appl. Mech. Engng*, to appear.

Comi, C. and Maier, G. (1990). Extremum theorem and convergence criterion for an iterative solution to the finite-step problem in elastoplasticity with mixed nonlinear hardening. *Eur. J. Mech. A/Solids* **9**(6), 563–585.

Corradi, L. (1978). On compatible finite element models for elastic plastic analysis. *Meccanica* **13**, 133–150.

Cruse, T. A. and Polch, E. Z. (1986). Elastoplastic BIE analysis of cracked plates and related problems. *Int. J. Num. Meth. Engng* **23**, 429–437 and 439–452.

De Donato, O. and Maier, G. (1972). Mathematical programming methods for the inelastic analysis of reinforced concrete frames allowing for limited rotation capacity. *Int. J. Num. Meth. Engng* **4**, 307–329.

Drucker, D. C. (1951). A more fundamental approach to plastic stress strain relations. *Proc. 1st U.S. Nat. Congr. Appl. Mech.* 487–491.

Gebbia, M. (1891). Formule fondamentali della statica dei corpi elastici. *Red. Circ. Matematico di Palermo* **5**, 320–323.

Halphen, B. and Nguyen, Q. S. (1975). Sur les matériaux standard généralisés. *J. de Mécanique* **14**, 39–63.

Krieg, R. D. and Krieg, D. B. (1977). Accuracies of numerical solution methods for the elastic-perfectly plastic model. *ASME, J. Pressure Vessel Techn.* **99**, 510–515.

Maier, G. (1983). On elastoplastic analysis by boundary elements. *Mech. Res. Comm.* **10**(1), 45–52.

Maier, G., Miccoli, S., Novati, G. and Sirtori, S. (1991). A Galerkin symmetric boundary-element method in elastoplasticity: formulation and implementation, to appear.

Maier, G. and Munro, J. (1982). Mathematical programming methods in engineering plastic analysis. *Appl. Mech. Rev.* **35**, 1631–1643.

Maier, G. and Nappi, A. (1984). On bounding post-shakedown quantities by the boundary value method. *Engineering Analysis* **1**, 223–229.

Maier, G., Novati, G. and Sirtori, S. (1989). Symmetric formulation of indirect boundary element method for elastic plastic analysis and relevant extremum properties. In *Boundary Element Methods in Applied Mechanics* (Edited by M. Tanaka and T. A. Cruse), pp. 215–224. Pergamon Press, Oxford.

Maier, G., Novati, G. and Sirtori, S. (1990). On symmetrization in boundary element elastic and elastic plastic analysis. In *Discretization Methods in Structural Mechanics*, Proc. IUTAM/IACM Symposium Vienna, Austria (1989) (Edited by G. Kuhn and H. Mang), pp. 191–200. Springer, Berlin.

Maier, G. and Polizzotto, C. (1987). A Galerkin approach to boundary element elastoplastic analysis. *Comp. Meth. Appl. Mech. Engng* **60**, 175–194.

Martin, J. B., Reddy, B. D., Grillin, T. B. and Bird, W. W. (1987). Applications of mathematical programming concepts to incremental elastic plastic analysis. *Engng Struct.* **9**, 171–176.

Mukherjee, S. (1982). *Boundary Element Methods in Creep and Fracture*. Applied Science, London.

Nguyen, Q. S. (1977). On the elastic plastic initial boundary value problem and its numerical integration. *Int. J. Num. Meth. Engng* **11**, 817–832.

Oden, J. T. and Brauchli, H. J. (1971). On the calculation of consistent stress distributions in finite element approximations. *Int. J. Num. Meth. Engng* **3**, 317–325.

Ortiz, M. and Martin, J. B. (1989). Symmetry preserving return mapping algorithms and incrementally extremal paths: a unification of concepts. *Int. J. Num. Method. Engng* **28**, 1839–1853.

Ortiz, M. and Popov, E. P. (1985). Accuracy and stability of integration algorithms for elastoplastic constitutive equations. *Int. J. Num. Meth. Engng* **21**, 1561–1576.

Parreira, P. and Guiggiani, M. (1989). On the implementation of the Galerkin approach in the boundary element method. *Comput. Struct.* **33**, 269–279.

Perego, U. (1988). Explicit backward difference operators and consistent predictors for linear hardening elastic–plastic constitutive laws. *Solid Mech. Archives* **13**(2), 65–102.

Polizzotto, C. (1988). An energy approach to boundary element method. *Comp. Meth. Appl. Mech. Engng* **69**, Part I: 167–184, Part II: 263–276.

Simo, J. C. (1991). Nonlinear stability of the time-discrete variational problem of evolution in nonlinear heat conduction, plasticity and viscoplasticity, to appear.

Simo, J. C. and Govindjee, S. (1990). Nonlinear B-stability and symmetry preserving return mapping algorithms for plasticity and viscoplasticity. *Int. J. Num. Meth. Engng* **31**, 151–176.

Simo, J. C., Kennedy, J. G. and Govindjee, S. (1988). Non-smooth multisurface plasticity and viscoplasticity. Loading/unloading conditions and numerical algorithms. *Int. J. Num. Meth. Engng* **26**, 2161–2185.

Sirtori, S. (1979). General stress analysis method by means of integral equations and boundary elements. *Meccanica* **14**, 210–218.

Sirtori, S., Maier, G., Novati, G. and Miccoli, S. (1991). A Galerkin symmetric boundary-element method in elasticity: formulation and implementation. *Int. J. Num. Meth. Engng* (to appear).

Smith, D. L. and Munro, J. (1978). On uniqueness in the elasto-plastic analysis of frames. *J. Struct. Mech.* **6**, 85–106.

Teixeira de Freitas, J. A. (1990). Mixed and hybrid symmetric formulations for the boundary integral method. *Eur. J. Mech. A Solids* **9**(1), 1–20.

Zienkiewicz, O. C., Kelly, D. W. and Bettess, P. (1977). The coupling of the finite element method and boundary solution procedures. *Int. J. Num. Meth. Engng* **11**, 355–375.